Jens Høyrup

# Practitioners – School Teachers – 'Mathematicians': The Divisions of Pre-Modern Mathematics and Its Actors

## Summary

The paper starts by looking at how 'practical' and 'theoretical' mathematics and their relation have been understood from the Greeks to Christian Wolff and by historians of mathematics from Montucla to recent days. Drawing on earlier work of mine, and on the giants on whose shoulders I (try to) stand, I then suggest a categorization of the mathematical knowledge types a historian has to deal with: the 'sub-scientific' type, carried by practitioners taught in an apprenticeship network; the 'scholasticized' type, taught supposedly for practice but in a 'scribal' school by masters whose own genuine practice is that of teaching; and the 'scientific' or theory-oriented type. In the end, the utility of this categorization is tried out on two specific cases.

Keywords: Knowledge types in mathematics; educational types; Old Babylonian 'algebra'; abbacus mathematics; historiography of mathematics.

In diesem Beitrag wird zunächst untersucht, wie ‚praktische' und ‚theoretische' Mathematik und ihre Verbindung aufgefasst wurden, von den antiken Griechen bis zu Christian Wolff und von Mathematikhistorikern von Montucla bis heute. Ausgehend von früheren Arbeiten von mir und von Riesen, auf deren Schultern ich stehe (oder zu stehen versuche), schlage ich anschließend eine Kategorisierung mathematischer Wissenstypen vor, mit denen sich ein Historiker auseinandersetzen muss. Den ‚sub-wissenschaftlichen' Typ verkörpern Praktiker, die als Lehrlinge von eigentlichen Praktikern unterrichtet wurden; der ‚Schulungs-Typ' wurde wohl auch für die Praxis gelehrt, allerdings vermutlich in ‚Schreiberschulen' von Meistern, deren eigene ‚Praxis' ausschließlich in der Lehre bestand; schließlich der ‚wissenschaftliche' theorie-orientierte Typ. Anhand von zwei Beispielen wird diese Einteilung am Ende überprüft.

Keywords: Wissenstypen in der Mathematik; Ausbildungstypen; altbabylonische ‚Algebra'; Abakus Mathematik; Historiographie der Mathematik.

John Steele, Mathieu Ossendrijver (eds.) | Studies on the Ancient Exact Sciences in Honor of Lis Brack-Bernsen | Berlin Studies of the Ancient World 44
(ISBN 978-3-9816384-5-5; URN urn:nbn:de:kobv:11-100246190) | www.edition-topoi.de

I suppose I have known Lis longer than any other contributor to this volume, years before any of us knew we would end up as historians – namely since she started university in 1964. By then, or at least a couple of years later, she intended to study physics, as I actually did. But even that was in the future – potential and never actualized future in Lis's case, as we know. Actually, physics was not what we spoke about by then; during her first year in mathematics, I was her instructor of algebra, so it was boolean logic, linear algebra and groups. Even though Lis was always sitting as far back as possible, I remember *where* she was sitting, close to the window. Already in first-year algebra, she was of course impressively bright. So, when we later ended up in our present-day pigeon-holes, we were in no doubt about each other.

Having never worked on astronomy (except when review editors have sent me something on the topic, not knowing that 'Babylonian mathematics' and 'Babylonian astronomy' are not only different topics but also on the whole as far in time from each other as Charlemagne from Churchill), I shall not contribute anything within Lis's own field. Instead I shall present a bird-eye's view of something I know better.

However, before approaching that subjectmatter, let me offer a personal note: I believe – but obviously cannot be sure about a matter of this kind – that my interest in practitioners' knowledge as an autonomous body goes back to the three years I taught physics at an engineering school some forty-five years ago, having thus moved away from the environment where I had met Lis. Among other things I remember one episode which to me has always illustrated the relationship between theoretician's knowledge and what is often (too often, I would argue) supposed to be its 'application'. Two colleagues – say, B and H – planned and held a course in electrodynamics for students of constructional engineering. H had been trained as an engineer himself, while B was a nuclear physicist. They were very good friends, and agreed upon most of what one can agree upon in this world. None the less, H one day complained to me that "B removes a Maxwell equation a year, but nothing changes!" Evidently, merely simplification of high theory was not what was needed in order to bridge the gap between the theoretician's and the engineering organization of knowledge.

## 1 Proto-historiography

Herodotos, followed by numerous other ancient Greek writers until Proclos, maintained that geometry began as (Egyptian) practice, and was later transformed into (Greek) theory; nothing was said by them about theory becoming in its turn the guide for the corresponding practice, although Hero and a few others tried to accomplish something like that (with modest impact outside the realm of war machines).

The standard view of the High Middle Ages – the epoch where the Latin Middle Ages had developed a scientific culture enabling them to form an opinion of their own in the matter and not just repeat what had already been repetition with Isidore – was not very different. In the introduction to the 'Adelard III' version of the *Elements*[1] we read that in the case of geometry, as in that of any other skill (*facultas*), *usus* not only preceded theory (*artificium*) but also continues to exist as the *exercitatio* of the skill; the main difference with respect to Antiquity is that the writer – himself certainly an *artifex* – demonstrates to have some interest in the practical *exercitatio*, as could reasonably be expected from someone who had Hugues de Saint-Victor in his intellectual luggage.

Though knowing *the field* of mathematics, perhaps both as theory/*artificium* and as a tool for practice / an *exercitatio*, neither Antiquity nor the medieval epoch was familiar with the figure of the *mathematician* in our sense of the word. At first, a μαθηματικός was a member of a branch of the Pythagorean movement; later in Antiquity, the *mathematicus* would mostly be an astrologer of the 'Chaldean' type; the teacher of the mathematical Liberal Arts – the closest we may perhaps come to a *professional* mathematician – would mostly be designated a *geometer*, while a μαθηματικός in the teacher's garb might teach any liberal or philosophical art. Aristotle does speak about the person who is engaged in mathematical argumentation as a μαθηματικός, but this is a personification of his ideal of epistemological autonomy of the various fields of knowledge, still no professional role. The Latin Middle Ages often did try to distinguish between the *matematicus*, that is, the astrologer, and the *mathematicus*, the one who practiced mathematics; but it would be difficult to find a person who primarily identified himself as a *mathematicus*.

Attitudes begin to change in the Renaissance. In a lecture on the mathematical sciences held in Padua by Regiomontanus in 1463/64 (the introduction to a series of lectures on al-Farghānī, printed by Schöner in 1537),[2] everything is seen in a (social as well as metatheoretical) top-down perspective. Mathematics is essentially *theory*, deriving its deserved high prestige, on one hand from its roots in classical Antiquity, on the other from its utility for philosophy and from its civic utility (which consists in procuring courtly pleasure). Much lower merit is ascribed to the applications[3] taught in the abbacus school (commercial computation, area calculation, etc.), and next to none to its use in material production. Whether these low-ranking applications are presumed to derive from theory is not clear.

Regiomontanus was ahead of his times, not only in the sense that he was a better mathematician than any contemporary in the Latin world but also in his attitudes to the character and role of mathematics (attitudes that he could only develop because of his mathematical insights and aptitudes); but a writer who is 'ahead of his times' is still

1 Ed. Busard 2001, 31–32.

2 Facsimile in Schmeidler 1972.

3 An inadequate term, since it presupposes that theory is 'applied to' (put upon) some practice; I use it for lack of a comprehensible one-word alternative.

bound to his times in many ways. A more mature expression of the conception of the relation between theory and practice that ripened during the later Renaissance is found in Vesalius's introduction to his *De humani corporis fabrica*.[4]

Vesalius, of course, discusses the medical art, not mathematics. This art, in his opinion, had been almost destroyed by the fact that responsibility for exerting it had been parceled out into three shares: that of the physician, the one who knows the principles of the art but does not know how to use a knife – or does not dare to lest his social standing might suffer; that of the pharmacist, who at least works under the guidance of the physician (that is, under the control of the medical faculty of universities); and that of the barber or surgeon, ignorant of everything according to Vesalius and therefore unable to do *adequately* that which in fact he does: use his hands. The art can only be restored to its former splendor if the three shares are once again united, *and 'the hand' brought under the control of the theoretically schooled physician*. In other words: practice – even the dirty practice of cutting and bloodletting – has to become applied science.

Vesalius, as is well known, inaugurated a period of rapidly progressing insights in anatomy. Medicine understood as the art of healing did not keep up with this progress in theory, but Vesalius had some sound justification for his claim. Slightly later we see a similar but stronger claim being made for mathematics by Petrus Ramus. Ramus, as is equally well known, wanted to avoid Euclid's 'Platonic error', the teaching of theory for theory's sake; but his alternative was an edition of the *Elements* where the proofs had been replaced by explanations of the utility of the single theorem. Theory should thus, as also requested by Vesalius, reform its mind and discard the mistaken fear of practical utility and dirty hands; but (reformed) theory should govern. In the historical introduction to Ramus's *Scholae mathematicae*[5] this view reveals its purely ideological character in the claim that the three famous great discoveries – the magnetic compass, gunpowder, and printing – were made in Germany because the mathematician Heinrich von Hessen had been forced to leave Paris in the 1380s and go to Vienna, thus inaugurating the blossoming of German mathematics; Ramus also wonders[6] that applied mathematics flourishes more in Italy than elsewhere in spite of the modest number of university chairs in mathematics, ignoring the existence of the abbacus school institution (*deliberately* ignoring it for sure, just as he deliberately ignores Stiefel from whom he copies wholesale though at the modest level he understands – probably indirectly but from authors like Jacques Peletier who do tell their debt to Stiefel).

In the sixteenth century, the 'mathematician' became a recognized social role, not least for those 'higher mathematical practitioners' who moved around the Italian courts;[7] Baldi's majestic *Vite de' matematici* illustrate the development.

4 Vesalius 1543, $2^r$–$2^v$.

5 Ramus 1569, 64–65.

6 Ramus 1569, 107.

7 Biagioli 1989.

What is most charitably characterized as Ramus's pipedream gradually materialized as reality over the next couple of centuries – first by the efforts of *Rechenmeister* like Tartaglia and Faulhaber to appropriate whatever Euclidean and Archimedean knowledge they might need (for their practice or for their social standing), afterwards in the interplay between these creators of new branches of mixed mathematics and mathematicians with scientific training and engaged in developing useful knowledge, for instance at the request of the Académie des Sciences. In his *Mathematisches Lexikon* from 1716, Christian Wolff recognizes that "*mathesis practica*, die ausübende Mathematick" as a category does not coincide with "*mathesis impura sive mixta*, die angebrachte Mathematick" – the latter being the application of mathematical understanding to "human life and nature", whether for the purpose of doing something *or* for obtaining theoretical insight.[8] He adds, however, that

> It is true that performing [*ausübende*] mathematics can be learned without reasoning mathematics; but then one remains blind in all affairs, achieves nothing with suitable precision and in the best way, at times it may occur that one does not find one's way at all. Not to mention that it is easy to forget what one has learned, and that that which one has forgotten is not so easily retrieved, because everything depends only on memory. Therefore all master builders, engineers, calculators, artists and artisans who make use of ruler and compass should have learned sufficient reasons for their doings from theory: this would produce great utility for the human race. Since, the more perfect the theory, the more correct will also every performance be.[9]

After the creation of the École Polytechnique and its nineteenth-century emulations there was no longer any need to repeat this protestation. For pragmatic reasons, Wolff's distinction between the 'practical' and the 'mixed' could be discarded – as it was already discarded in the names given by Gergonne and Crelle to their journals, respectively *Annales de mathématiques pures et appliquées* and *Journal für reine und angewandte Mathematik*.

## 2 Historiography

Modern historiography of mathematics begins, we might say, with the generations from Montucla and Cossali to Libri and Nesselmann. These were still close to the victory of the 'Vesalian' subordination of practice under reformed theory, and furthermore

8 Wolff 1716. On pp. 866–867 Wolff observes that "everything in mathematics beyond arithmetic, geometry and algebra [which constitute his 'pure mathematics'] belongs to accommodated mathematics". As everywhere in the following where no other translator is identified, the translation is mine.

9 Wolff 1716, 867.

brought up mathematically before the triumphs of the 'new' pure mathematics inaugurated by Cauchy, Abel, etc. Finally, they were hungry for sources of any kind. No wonder hence that their attitudes would still have some of their roots in the situation delineated by Wolff. Montucla, when telling in his second edition[10] about Ottoman, Arabic, Persian, and Indian mathematics, actually applies what in one of the current meanings of that word can be characterized as an *ethnomathematical* perspective, describing (briefly) teaching practices as well as the uses of mathematics and computation in general social life.[11]

However, the interest in practical mathematics did not die with their generations. When dealing with pre-Modern mathematics, historians like Boncompagni, D. E. Smith, Tropfke, Karpinski, and Vogel would still pay much attention to sources that had their roots in practice. At least as a rule, they abstained from using the term 'mathematicians' about the originators of what several of them termed 'school mathematics' or 'elementary mathematics'. Given the sources they relied on,[12] neither designation was mistaken; but they express a belief in the unity of the mathematical genres that agrees with Wolff's ideal (and with the perspective of their own times) but not – as I shall argue – with the social reality of pre-Modern mathematics.

Around 1930, the perspective changed.[13] History of mathematics came to be understood as the history of the *mathematics of mathematicians*, and mathematicians tended to be defined in post-Cauchy-Abel terms. In part that was a consequence of the disappearing interest in European medieval mathematics, on which next to nothing was published between 1920 and 1948.[14] But this explanation from the object of the historian is partial at best: in the 1920s and the early 1930s, the appearance of two good editions of the Rhind Mathematical Papyrus and the publication of the Moscow Mathematical Papyrus spurred some further publication activity; from the late 1920s onwards, the Babylonian mathematical texts were cracked and published, which had a great impact, not least through the acceptance of Neugebauer's thesis about the descent of Greek 'geometric algebra' from Babylonian 'algebra'. However, in the perspective of the epoch, even

10 Montucla and Lalande 1799–1802, in particular Vol. I, 397–402, but also elsewhere.

11 Cf. D'Ambrosio 1987; Mesquita, Restivo, and D'Ambrosio 2011.

12 Namely, manuscripts and printed works. Montucla, when making his proto-ethnomathematics, had relied on ethnographic informants (diplomats and other travelers), and elsewhere uses his direct acquaintance with practitioners to supplement what he can document from written sources. But the historians of mathematics of the following 150 years, like other historians from von Ranke's century, relied on *documents*.

13 Given, for instance, that Vogel lived and worked until 1985 it goes by itself that this statement is an extremely rough approximation to *wie es eigentlich gewesen*, permissible only in the context of an introductory discussion.

14 Most of the few publications that did appear are from Karpinski's hands. If these are excluded, the general absence of interest in medieval Latin and vernacular European mathematics becomes even more striking.

Babylonian mathematics came to be understood as the product of 'Babylonian mathematicians'.[15] Moreover, even the historiography of Early Modern mathematics tended to turn away from the applications of mathematical theory and to concentrate on 'real' mathematics.

## 3 Missed opportunities

Two events should be mentioned at this point, not because they affected the historiography of mathematics but rather because it might seem strange that they did not.

The first is the renowned intervention of Soviet scholars at the London Congress of the History of Science in 1931.[16] Within the historiography of science, Boris Hessen's paper on "The Social and Economic Roots of Newton's *Principia*" was indubitably the one that had the strongest impact. By way of J. D. Bernal's reception and ensuing successful campaign for the implementation of *science policy*, Bukharin's paper on "Theory and Practice from the Standpoint of Dialectical Materialism" and M. Rubinstein's presentation of the "Relations of Science, Technology, and Economics under Capitalism and in the Soviet Union" were probably those that were most consequential.

Hessen's paper was written under conditions which his audience did not know about, and carried a subtle message that it missed.[17] Bukharin shared Hessen's fate not only in life (both fell victims to Stalin's purges in 1938) but also as regards his London paper. As observed by I. Bernard Cohen, "Bukharin's piece remains impressive today [c. 1989] to a degree that Hessen's is not".[18] But that went largely unnoticed in 1931.

Bukharin discusses the relation between theory and practice both from an epistemological and from a sociological point of view. On the first account he emphasized that knowledge comes not from pure observation but from intervention in the world – which may not go beyond what he cites from Marx, Engels, and Lenin though certainly beyond what his audience knew about what these authors had said, and which in any case had to wait for Mary Hesse and Thomas Kuhn before it was accepted outside Marxist circles. On the second account – the one that is relevant for our present purpose – he emphasized the complexity and historically conditioned mutability of the relation between knowledge and practice, as well as the changing ways in which different types of knowledge are distributed between carriers with different social roles.

15 I do not remember Neugebauer to have employed the expression; in Neugebauer 1934, 125 n. 1 he rejects the notion of 'mathematicians' very explicitly with reference to Egypt; but it was used by Thureau-Dangin – e.g., Thureau-Dangin 1938, xxxviii – and afterwards by various other authors, although most would speak simply of 'the Babylonians'.

16 The Soviet contributions, printed already at the moment, were reprinted in 1971 as *Science at the Cross Roads* (Bukharin et al. 1971).

17 See Graham 1993, 143–151.

18 Graham 1993, 141.

As alluded to, Bukharin's subtleties proved too subtle for the Western audience, and had no impact.[19] Even Joseph Needham, who later was to make the non-trivial interplay between 'clerks and craftsmen' a favorite theme of his, only saw Bukharin's paper as "in its way a classical statement of the Marxist position".[20] Needham instead received his impulse from the second of the above-mentioned events: Zilsel's paper on "The Sociological Roots of Science"[21] (as well as other papers by the same author).

Recent work on Zilsel's *Nachlaß*[22] shows that this and other papers of his from the same period belong within a larger metatheoretical project that never materialized as such. As it stands and on its own, the paper argues that the discussion about *the* root of the new science of the late sixteenth and the seventeenth century – whether scholastic thought, Humanism, or the knowledge of engineers like Leonardo da Vinci – is mistaken, since it was *the interplay* between natural philosophers belonging to the scholastic tradition, trained Humanists, and 'higher artisans' that made possible the breakthrough.

Needham was not the only historian *of science* to be impressed by Zilsel's paper, which (like Hessen's article) indeed called forth a number of other publications either taking up the thesis or explicitly arguing against it. Strangely, however, no historian of mathematics seems to have addressed the questions whether Zilsel's thesis might apply *mutatis mutandis* to the revolution in early Modern mathematics.[23] Initially this non-reaction was perhaps not so strange – at the time, and for long, historians of mathematics saw in the most important group of 'higher artisans' of relevance for the question (the Italian abbacus masters) nothing but not very competent *vulgarisateurs* of Leonardo Fibonacci (if they happened to know at all about their existence); ascribing to such people a stimulating influence was more than could be expected from historians concentrating on the mathematics of (great) mathematicians.[24]

19 They may also have been too subtle for his fellow-countrymen, but until Bukharin's rehabilitation in 1988 these had other reasons not to get too close. For decades, the points of view expressed by the Soviet delegation at the London Congress could only be discussed in the Soviet Union as filtered through Bernal's not very sophisticated reception.

20 Bukharin et al. 1971, ix.

21 Zilsel 1942.

22 Raven and Krohn 2000.

23 At least not before I organized an international workshop on the theme "'Higher artisans', Humanism and the University Tradition. The Zilsel thesis reconsidered in relation to the Renaissance transformation of mathematics" in 1998 – but even then it did not really happen (Paolo Rossi, who would probably have understood, was forced to cancel his participation). In consequence, I had to take up the theme on my own in Høyrup 2011.

24 Karpinski's closing commentary to Jacopo da Firenze's abbacus treatise, though preceding Zilsel's paper, is characteristic of the attitude that prevailed afterwards (Karpinski 1929, 177): "[the early fourteenth-century] treatise by Jacob of Florence, like the similar [late fifteenth-century] arithmetic of Calandri, marks little advance on the arithmetic and algebra of Leonard of Pisa. The work indicates the type of problems which continued current in Italy during the thirteenth to the fifteenth and even sixteenth centuries, stimulating abler students than this Jacob to researches which bore fruit in the sixteenth century in the achievements of Scipione del Ferro, Ferrari, Tartaglia, Cardan and Bombelli." As we see, Fibonacci, Jacopo, Calandri, and Bombelli are supposed to belong on *the same* branch, although part of it has undergone some degeneration.

No 'event' but a process has been the increasing awareness within the history of technology that pre-nineteenth-century technical knowledge, including knowledge leading to technical innovation, cannot be adequately described as 'applied science'. Even this process has left fewer traces in the historiography of mathematics than it should perhaps have done.

## 4 'Popular' or 'sub-scientific'

In spite of the invitations of Bukharin and Zilsel it thus remained common, to the extent the mathematics of medieval and other pre-Modern practitioners was at all taken into account and seen as a different body than that of the 'scientific' traditions, to characterize it as 'popular' or 'folk'. I still did so myself in in my contribution to the Sarton Centennial Conference[25] when discussing the roots for those aspects of the Islamic mathematical corpus which lexicographers like al-Nadīm do not trace to the Greeks but treat as anonymous traditions or fail to mention.

But evidently neither the use of the 'Hindu numerals' nor trigonometry were known at the time by 'people' in general; these kinds of supposedly 'popular' knowledge were carried by narrow social groups and thus certainly constituted *specialists' knowledge*. In consequence I began speaking of these sources and the traditions to which they belonged as 'sub-scientific', first in passing,[26] then more analytically.[27] On occasion of Bukharin's centennial I elaborated this discussion,[28] emphasizing the oral cultural type of the carrying environment and pointing (i) to the function of (what has come to be misnamed) 'recreational problems'[29] as 'neck riddles' that display appurtenance to a particular craft carrying a particular body of know-how, (ii) to the possibility to use these problems (as eventually adopted into cultures leaving written sources, thereby becoming properly 'recreational') as index fossils allowing us to trace an oral culture that in the nature of things is not directly documented in writing.[30]

In this paper I still used the term 'sub-scientific' about scribal as well as non-literate practitioners' mathematics, singling out the former type as nothing but a sub-category. Schools – even pre-Modern schools teaching practical mathematics – certainly vary in

25 Høyrup 1984.

26 Høyrup 1986.

27 Høyrup 1987.

28 Published as Høyrup 1990b.

29 More precisely: the problems become 'recreational' when adopted into literate culture; the term is only a misnomer in relation to their original function.

30 Also in the later 1980s, David King investigated the astronomy of Islamic legal scholars and pointed out that it was distinct from the astronomy of mathematicians. He used the term 'folk astronomy' but left no doubt that it was the astronomy of the 'craft' of legal scholars. See the papers contained in King 1993.

character, and can be argued to constitute a pluri-dimensional continuum merging gradually into oral apprenticeship teaching on one side; but it is also difficult, even in several pre-Modern settings, to make a totally clean cut between schools teaching for practice and schools teaching 'scientific' mathematics.[31] I would therefore now distinguish between the *sub-scientific* knowledge type, carried by practitioners taught in an apprenticeship network; the *'scholasticized'* or *scribal* practitioners' knowledge type, communicated in a school by masters whose own genuine practice is that of teaching, not the practical use of the knowledge they teach; and the *'scientific'* or *theory-oriented* type, the one to which historians of mathematics have dedicated most of their efforts – keeping in mind that these are fuzzy categories understood through ideal types functioning as navigational guides rather than classificatory boxes.[32]

## 5 Applications of the categorization

Networks of categories constitute an instance of formal knowledge (albeit of the most primitive kind). Their utility thus depends on their ability to create order in the tangle of real-world phenomena – those from which they were derived in the first instance through a process of abstraction (that should be the easier but still the obvious first test) as well as others that did not intervene when they were constructed (not necessarily quite as easy). I shall look at one instance of each kind.

When speaking for the first time of a 'sub-scientific tradition' in 1986 I referred to the tradition that linked Old Babylonian 'algebra' to the area riddles in Abū Bakr's *Liber mensurationum*. Some years later,[33] I also voiced a suspicion that the problem BM 13901 #23 (dealing with a square, for which the sum of *the four* sides and the area is given) was "a surveyors' recreational problem, maybe from a tradition that was older than – perhaps even a source for – Old Babylonian scribal school 'algebra'"; I also observed the family resemblance of the configuration used in the solution with one of al-Khwārizmī's proofs. However, by then I had to leave both matters there.

Over the following years, being alerted to the stylistic peculiarities that might characterize fresh borrowings from an oral tradition as well as to those that should correspond to transmission within a stable school environment (and being in general stimulated to be sensitive to stylistic detail and not only to so-called 'mathematical substance')

31 See, for late Greco-Roman Antiquity, Cuomo 2000.

32 Cf. Høyrup 1997. It might be useful to distinguish a fourth type, the 'deuteronomic' teaching of theory petrified into and taught in school as a dignified tradition – the shape in which most of the students taught scientific mathematics have encountered their Euclid since two thousand years; cf. Netz 1998. But since my topic is the relation between mathematical practice and mathematical theory I shall not pursue this theme at present.

33 Høyrup 1990a, 275.

I was able (that is at least my own opinion) to put on a firmer footing than done before the claim that Old Babylonian 'algebra' and Euclidean 'geometric algebra' (both 'so-called') were connected, and to demonstrate also that the geometric riddles of Arabic *misāḥa* treatises as well as al-Khwārizmī's geometric proofs for the basic *al-jabr* procedures belonged within the same network. Moreover I could argue (still of course in my own opinion) that the Old Babylonian 'algebraic' school discipline built upon original borrowings from the neck riddles of a lay surveyors' environment, and that this environment and its riddles, not the tradition of scholar-scribes, was responsible for the transmission of the inspiration to later times.

Since I have described this analysis and its outcome at length elsewhere,[34] I shall not go into further detail, but turn instead to a historical phase which I started looking seriously at some fifteen years ago: the Italian abbacus school of the late Middle Ages and the Renaissance, and its relation to Leonardo Fibonacci.

Karpinski, who was one of the first to describe the stylistic peculiarities of an abbacus treatise (Jacopo da Firenze's above-mentioned *Tractatus algorismi* from 1307, in Tuscan in spite of the Latin title, and written in Montpellier), though quite aware of its deviations from what can be read in Fibonacci's *Liber abbaci*, still appraised its contents as if it was only a station on the road from Fibonacci to Scipione del Ferro (see note 24). At the moment little systematic work had been done on abbacus material,[35] but things did not change even when Gino Arrighi and his pupils had published an appreciable number of manuscripts. Wholly in Karpinski's vein, Kurt Vogel stated that Cowley's description of the Columbia ms X 511 A1 3 had been important because it "filled a lacuna between Leonardo da Pisa's *Liber abbaci* and Luca Pacioli's *Summa*".[36] Even sharper are the formulations of those who have worked most intensely on the material – thus Warren Van Egmond, according to whom all abbacus writings "can be regarded as [...] direct descendants of Leonardo's book";[37] and Raffaella Franci and Laura Toti Rigatelli, according to whom "the abacus schools had risen to vulgarize, among the merchants, Leonardo's mathematical works".[38] More recently, Elisabetta Ulivi – probably the scholar who has worked most in depth on the social history of the abbacus environment – has expressed the view that the abbacus treatises "were written in the vernaculars of the various regions, often in Tuscan vernacular, taking as their models the two important works of Leonardo Pisano, the *Liber abaci* and the *Practica geometriae*".[39]

34 Most extensively in Høyrup 2001 and Høyrup 2002, 362–417.

35 Karpinski (1910/1911) describes another abbacus algebra, and Cowley (1923) analyzes a whole treatise. During the nineteenth century a number of excerpts had been published by Libri, Boncompagni, and others, but no coherent descriptions of whole treatises (nor *a fortiori* of the category as such) had appeared.

36 Vogel 1977, 3.

37 Van Egmond 1980, 7.

38 Franci and Toti Rigatelli 1985, 28.

39 Ulivi 2002, 10. Similarly in more recent publications.

All of these, I would claim, have fallen victims to the 'syndrome of The Great Book', the conviction that every intellectual current has to descend from a *Great Book* that is *known to us* at least by name and fame – the same conviction that made those who objected to Neugebauer's proposed transmission observe that no Greek would have bothered to read the Babylonian clay tablets, and induced many of those who have discussed the possible borrowing of Indian material into Arabic algebra to believe that inspiration had to come from the writings of an Āryabhaṭa or a Brahmagupta.

Already Karpinski had noticed that Jacopo's algebra has no problems in common with the *Liber abbaci*. Reading of the whole treatise shows it to have no single problem, algebraic or otherwise, in common with the Great Book, but to contain on the other hand numerous problems belonging to classes that are also present in that Book.[40] Some of these belong to the cluster of problems that are found in ancient and medieval sources "from Ireland to India", as Stith Thompson says about the 'European folktale'[41] – and even in the Chinese *Nine Chapters*. This cluster of problems usually going together was apparently carried by the community of merchants traveling along the Silk Road[42] and adopted as 'recreational problems' by the literate in many places; it is thus a good example of a body of sub-scientific knowledge influencing school knowledge systems in many places and an illustration of the principle that it is impossible to trace the 'source' for a particular trick or problem in a situation where "the ground was wet everywhere".[43]

Other problem types are shared with Fibonacci but not diffused within the larger area (or diffused within a different larger area that may coincide with the Arabic network of sea trade from the Indian Ocean to the Mediterranean). Moreover, Jacopo employs a range of set phrases ("et così se fanno tucte le simile ragioni", "se ci fosse data alcuna ragione", etc.) that also turn up copiously in other abbacus writings as well as in similar writings from the Provençal-Catalan and the Castilian area[44] – and also, but on so rare occasions that they seem to represent slips, in Fibonacci's text.

A slightly earlier Umbrian abbacus treatise (Florence, Riccardiana ms. 2404, from c. 1290)[45] claims in its title to be "according to the opinion" of Fibonacci. Analysis of the text shows this claim to be misleading.[46] Everything basic in the treatise is as different from what we find in the *Liber abbaci* as is Jacopo's *Tractatus* (and characterized by the presence of the same set phrases); but the writer borrows a number of sophisticated problems from Fibonacci, often demonstrably without understanding even as

40 Cf. Høyrup 2007, which contains an edition and English translation of the work.

41 Thompson 1946, 13.

42 Some of the traveling problems deal precisely with bits of this web of caravan and sea routes extending from China to Cadiz, and no other network (however open-ended) existed that ranged so widely.

43 Høyrup 1987, 290.

44 See Sesiano 1984, a description of the Pamiers algorism; Malet 1998, an edition of Francesc Santcliment's *Summa de l'art d'aritmetica*; and Caunedo del Potro and Córdoba de la Llave 2000, an edition of the Castilian *Arte del alguarismo*.

45 Arrighi 1989.

46 Høyrup 2005.

much as the notation of his source. Obviously, Fibonacci had already become a kind of culture hero (modern historians are not the first to fall victims to the syndrome of The Great Book), and the borrowings serve as embellishment beyond the ordinary teaching matters.

From combination of these pieces of evidence it becomes obvious that Jacopo's as well as the Umbrian treatise refer to an environment spread out in all probability over much of the Romance-speaking Mediterranean region, already in possession of elementary vernacular literacy and probably based in some kind of school teaching similar to the Italian abbacus school but with at most tenuous ties to the world of university scholars. It also becomes clear that already Fibonacci had drawn part of his inspiration for the *Liber abbaci* from this environment, whose existence thus antedates 1200 (or at the very least 1228).

Analysis of Jacopo's algebra chapter and comparison with Arabic algebraic writings suggests that it is ultimately drawn from another level of Arabic algebra than that of the Great Books of al-Khwārizmī, Abū Kāmil, Ibn al-Bannāʾ, etc. It seems likely – but for the time being cannot be conclusively established – that the just-mentioned school environment was not restricted to the Romance-speaking area but also reached into (and probably came from) a similar environment in the Arabic Mediterranean teaching *muʿāmalāt*-mathematics (even Arabic merchants must have learned their mathematics somewhere, including the use of the rule of three to which already al-Khwārizmī had dedicated the "Chapter on *muʿāmalāt*" of his *Algebra*.[47] That school in Bejaïa in which Fibonacci tells to have spent "some days" learning the *studium abbaci*[48] is likely to have been such a school (the alternative, a mosque school, is not plausible).[49]

Though in all probability a descendant of a school environment that had inspired both Fibonacci and Jacopo, the mature Italian abbacus school of the fourteenth and fifteenth century developed characteristics that are not likely to have been present before 1310 – characteristics that appear to have depended on the market competition between abbacus masters for jobs and pupils. Both the Umbrian abbacus and Jacopo's treatise make mathematical mistakes from time to time – but they abstain from mathematical fraud. Already within the first two decades after Jacopo's writing of (what is in all probability) the first Italian vernacular algebra, on the other hand, abbacus treatises

47 Ed. Rosen 1831, Arabic 48.

48 Ed. Boncompagni 1857, 1.

49 Some of the formulations in Jacopo's discussion of metrologies are strikingly similar to what we find in Aḥmad ibn Thabāt's *Reckoners' Wealth* from c. 1200 (*Ġunyat al-ḥussāb*, ed. Rebstock 1993), which however both surpasses what it would be reasonable to teach to practical reckoners (e.g., Euclidean geometric definitions) and offers too little training for these; but ibn Thabāt was a scholar who taught law as well as *ḥadīth* and *ʿilm al-ḥisāb* at the Niẓāmīya madrasah (Rebstock 1993, x), and thus wrote a scholarly book about practitioners' mathematics, no textbook for the training of merchant youth. Apart from his own intellectual pleasure, he may have been motivated by what (for instance) a judge had to understand about all domains of practical computation.

begin to present blatantly false rules for irreducible equations of the third and fourth degree – not easily unmasked by competitors, however, because the examples are always chosen so as to lead to 'solutions' containing radicals. Only at a moment when abbacus-trained writers like Luca Pacioli began moving on the interface between the Humanist-courtly and the scholastic-scholarly areas[50] was the fraud exposed – and only then was there space for del Ferro's genuine solution to contribute to the revolution in mathematics (in good agreement with the Zilsel thesis, we might say).

Italian abbacus mathematics is thus not to be understood as an activity bridging one Great Book (the *Liber abbaci*) and another one (e.g., Cardano's *Ars magna*) but as a distinct undertaking, carried neither by scholarly mathematicians nor by a purely oral culture, yet having most of its ultimate roots in an environment close to the latter type, and giving eventually important stimuli to the further development of scientific mathematics. I shall permit myself to claim that the categorization suggested above is fruitful in opening our eyes to evidence in the sources that has so far been overlooked, and thus allows us to attain better understanding of the real historical process. At the same time the example demonstrates that a seemingly simple category ('schools') covers phenomena of widely different character, held together mainly by being neither orally based nor 'scientific' in ambition.[51]

50 That Luca moved in this zone is quite obvious, e.g., both from his preface to the *De divina proportione* (ed. trans. Winterberg 1889, 17–35) and from his publication of the Campanus version of the *Elements*.

51 This point could be sharpened if the abbacus school were contrasted, e.g., with the Old Babylonian scribe school, which eliminated mathematical fraud (namely, mock solutions) from its sub-scientific heritage. Analysis of what happens to a specific problem type, e.g., the 'hundred fowls,' might highlight the difference between the genuinely sub-scientific and the abbacus-school style.

# Bibliography


**Arrighi 1989**
Gino Arrighi, ed. *Maestro Umbro (sec. XIII), Livero de l'abbecho. (Cod. 2404 della Biblioteca Riccardiana di Firenze)*. Bollettino della Deputazione di Storia Patria per l'Umbria 86. 1989, 5–140.

**Biagioli 1989**
Mario Biagioli. "The Social Status of Italian Mathematicians, 1450–1600". *History of Science* 27.1 (1989), 41–95. DOI: 10.1177/007327538902700102.

**Boncompagni 1857**
Baldassare Boncompagni, ed. *Scritti di Leonardo Pisano, matematico del secolo decimoterzo, I: Il Liber abaci di Leonardo Pisano*. Roma: Tipografia delle Scienze Matematiche e Fisiche, 1857.

**Bukharin et al. 1971**
Nikolaï Ivanovich Bukharin et al. *Science at the Cross Roads: Papers Presented to the [Second] International Congress of the History of Science and Technology, Held in London from June 29th to July 3rd, 1931, by the Delegates of the USSR*. With an add. forew. by Joseph Needham and an intro. by P. G. Werskey. Social History of Science 23. London: Frank Cass, 1971.

**Busard 2001**
Hubert L. L. Busard, ed. *Johannes De Tinemue's Redaction of Euclid's Elements, the So-Called Adelard III Version: Vol. I, Introduction, Sigla and Descriptions of the Manuscripts, Editorial Remarks, Euclides, 'Elementa'; Vol. II, Conspectus siglorum, apparatus criticus, addenda*. Boethius 45.1.–45.2. Stuttgart: Franz Steiner, 2001.

**Caunedo del Potro and Córdoba de la Llave 2000**
Betsabé Caunedo del Potro and Ricardo Córdoba de la Llave, eds. *El arte del alguarismo: un libro castellano de aritmética comercial y de ensayo de moneda del siglo XIV (Ms. 46 de la Real Colegiato de San Isidoro de León)*. Estudios de historia de la ciencia y de la técnica, n. 16. Salamanca and Valladolid: Junta de Castilla y León, Consejeria de Educación y Cultura, 2000.

**Cowley 1923**
Elisabeth Buchanan Cowley. "An Italian Mathematical Manuscript". In *Vassar Mediæval Studies, by Members of the Faculty of Vassar College*. Ed. by C. F. Fiske. New Haven: Yale University Press, 1923, 379–405. URL: http://hdl.handle.net/11375/14716 (visited on 17/7/2017).

**Cuomo 2000**
Serafina Cuomo. *Pappus of Alexandria and the Mathematics of Late Antiquity*. Cambridge Classical Studies. Cambridge (UK): Cambridge University Press, 2000.

**D'Ambrosio 1987**
Ubiratan D'Ambrosio. *Etnomatematica: Raizes Socio-Culturais da Arte ou Tecnica de Explicar e Conhecer*. Campinas, São Paulo: Universidad de Campinas, 1987.

**Franci and Toti Rigatelli 1985**
Raffaella Franci and Laura Toti Rigatelli. "Towards a History of Algebra from Leonardo of Pisa to Luca Pacioli". *Janus* 72 (1985), 17–82.

**Graham 1993**
Loren R. Graham. *Science in Russia and the Soviet Union: A Short History*. Cambridge Studies in the History of Science. Cambridge (UK): Cambridge University Press, 1993.

**Høyrup 1984**
Jens Høyrup. *Formative Conditions for the Development of Mathematics in Medieval Islam*. Roskilde: Roskilde University Centre, Institute of Educational Research, Media Studies and Theory of Science, 1984.

**Høyrup 1986**
Jens Høyrup. "Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: On the Origins of Islamic Algebra". *Erdem* 2 (1986), 445–484.

**Høyrup 1987**
Jens Høyrup. "The Formation of 'Islamic Mathematics': Sources and Conditions". *Science in Context* 1.1 (1987), 281–329. DOI: 10.1017/S0269889700000399.

**Høyrup 1990a**
Jens Høyrup. "Algebra and Naive Geometry: An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought". Pt. I and II. *Altorientalische Forschungen* 17 (1990), 27–69, 262–354.

**Høyrup 1990b**
Jens Høyrup. "Sub-Scientific Mathematics: Observations on a Pre-Modern Phenomenon". *History of Science* 28.1 (1990), 63–87. DOI: 10.1177/007327539002800102.

**Høyrup 1997**
Jens Høyrup. "Mathematics, Practical and Recreational". In *Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures*. Ed. by H. Selin. Dordrecht: Kluwer Academic, 1997, 660–663. DOI: 10.1007/978-1-4020-4425-0_8747.

**Høyrup 2001**
Jens Høyrup. "On a Collection of Geometrical Riddles and Their Role in the Shaping of Four to Six 'Algebras'". *Science in Context* 14 (2001), 85–131. DOI: 10.1017/S0269889701000047.

**Høyrup 2002**
Jens Høyrup. *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin*. Sources and Studies in the History of Mathematics and Physical Sciences. New York, Berlin, and Heidelberg: Springer Science, 2002. DOI: 10.1007/978-1-4757-3685-4.

**Høyrup 2005**
Jens Høyrup. "Leonardo Fibonacci and Abbaco Culture: A Proposal to Invert the Roles". *Revue d'Histoire des Mathématiques* 11.1 (2005), 23–56. URL: https://eudml.org/doc/252091 (visited on 17/7/2017).

**Høyrup 2007**
Jens Høyrup. *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture*. Science Networks, Historical Studies 34. Basel: Birkhäuser, 2007. DOI: 10.1007/978-3-7643-8391-6.

**Høyrup 2011**
Jens Høyrup. "The Fifteenth-Seventeenth Century Transformation of Abbacus Algebra: Perhaps – Though Not Thought of by Edgar Zilsel and Joseph Needham – the Best Illustration of the 'Zilsel-Needham Thesis'". *Summer School on the History of Algebra, Institute for the History of the Natural Sciences, Chinese Academy of Science, 1–2 September 2011*. Preprint (2011). URL: http://ruc.dk/~jensh/Publications/2011%7ba%7d_15th-17th-c.%20transformation%20of%20abbacus%20algebra.PDF (visited on 17/7/2017).

**Karpinski 1910/1911**
Louis Charles Karpinski. "An Italian Algebra of the Fifteenth Century". *Bibliotheca mathematica (3rd series)* 11.3 (1910/1911), 209–219. URL: http://magteca-fi-ese.inera.it/unifi/opac/unifi/scheda.jsp?pid=mag:6451 (visited on 17/7/2017).

**Karpinski 1929**
Louis Charles Karpinski. "The Italian Arithmetic and Algebra of Master Jacob of Florence, 1307 (2 Tab.)" *Archeion* 11.2-3 (1929), 170–177. DOI: 10.1484/J.arch.3.341.

**King 1993**
David A. King. *Astronomy in the Service of Islam*. Variorum Collected Studies Series. Aldershot and Brookfield: Routledge and Variourum, 1993.

**Malet 1998**
Antoni Malet, ed. *La Summa de l'art d'Aritmètica de Francesc Santcliment (1482)*. Textos d'Historia de la Ciencia 1. Vic, Barcelona: Eumo Editorial, 1998.

**Mesquita, Restivo, and D'Ambrosio 2011**
Mônica Mesquita, Sal Restivo, and Ubiratan D'Ambrosio, eds. *Asphalt Children and City Streets: A Life, a City and a Case Study of History, Culture, and Ethnomathematics in São Paulo*. Rotterdam, Boston, and Taipei: SensePublishers, 2011. DOI: 10.1007/978-94-6091-633-5.

**Montucla and Lalande 1799–1802**
J.-F. [i.e. Jean-Étienne] Montucla and Joseph Jérôme Lefrançais Lalande. *Histoire des mathématiques*. 4 vols. Tomes III et IV achevés et publiés par Jérôme de La Lande. Paris: Henri Agasse, 1799–1802. URL: http://catalogue.bnf.fr/ark:/12148/cb33999114b (visited on 17/7/2017).

**Netz 1998**
Reviel Netz. "Deuteronomic Texts: Late Antiquity and the History of Mathematics". *Revue d'Histoire des Mathématiques* 4.2 (1998), 261–288. URL: https://eudml.org/doc/252027 (visited on 17/7/2017).

**Neugebauer 1934**
Otto E. Neugebauer. *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften 1: Vorgriechische Mathematik*. Berlin: Julius Springer, 1934.

**Ramus 1569**
Petrus Ramus. *Scholarum mathematicarum libri unus et triginta*. Basel: Eusebius Episcopius, 1569.

**Raven and Krohn 2000**
Diederick Raven and Wolfgang Krohn. "Edgar Zilsel: His Life and Work (1891–1944)". In *The Social Origins of Modern Science*. Ed. by E. Zilsel, D. Raven, W. Krohn, and R. S. Cohen. Boston Studies in the Philosophy and History of Science 200. Dordrecht: Kluwer Academic, 2000, xxi–lxii. DOI: 10.1007/978-94-011-4142-0.

**Rebstock 1993**
Ulrich Rebstock. *Die Reichtümer der Rechner (Ġunyat al-ḥussāb) von Aḥmad b. Ṯabāt (gest. 631/1234): Die Araber – Vorläufer der Rechenkunst*. Beiträge zur Sprach- und Kulturgeschichte des Orients 32. Walldorf-Hessen: Verlag für Orientkunde Dr. H. Vorndran, 1993.

**Rosen 1831**
Frederic Rosen, ed. and trans. *The Algebra of Muhammad ben Musa*. London: The Oriental Translation Fund, 1831.

**Schmeidler 1972**
Felix Schmeidler, ed. *Joannis Regiomontani opera collectanea. Faksimile-Drucke von neun Schriften Regiomontans und einer von ihm gedruckten Schrift seines Lehrers Purbach*. Milliaria 10.2. Osnabrück: Otto Zeller, 1972.

**Sesiano 1984**
Jacques Sesiano. "Une Arithmétique médiévale en langue provençale". *Centaurus* 27.1 (1984), 26–75. DOI: 10.1111/j.1600-0498.1984.tb00753.x.

**Thompson 1946**
Stith Thompson. *The Folktale*. New York: The Dryden Press, 1946.

**Thureau-Dangin 1938**
François Thureau-Dangin. *Textes Mathématiques Babyloniens*. Publications de la Société Orientale Ex Oriente Lux 1. Leiden: Brill, 1938.

**Ulivi 2002**
Elisabetta Ulivi. "Benedetto da Firenze (1429–1479), un maestro d'abaco del XV secolo. Con documenti inediti e con un'Appendice su abacisti e scuole d'abaco a Firenze nei secoli XIII–XVI". *Bollettino di Storia delle Scienze Matematiche* 22.1 (2002), 3–243. URL: https://www.libraweb.net/articoli.php?chiave=200209201&rivista=92 (visited on 17/7/2017).

**Van Egmond 1980**
Warren Van Egmond. *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abbacus Manuscripts and Printed Books to 1600*. Monografia 4. Firenze: Istituto e Museo di Storia della Scienza, 1980.

**Vesalius 1543**
Andreas Vesalius. *De humani corporis fabrica libri septem*. Basel: Johann Oporinus, 1543. URL: http://www.bvh.univ-tours.fr/Consult/index.asp?numfiche=56 (visited on 17/7/2017).

**Vogel 1977**
Kurt Vogel. *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A13)*. Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik, Reihe C, Quellentexte und Übersetzungen, Nr. 33. Munich: Forschungsinst. d. Dt. Museums, 1977.

**Winterberg 1889**
Constantin Winterberg, ed. *Fra Luca Pacioli, 'De Divina Proportione': Die Lehre vom Goldenen Schnitt, nach der Venezianischen Ausgabe vom Jahre 1509 neu herausgegeben, übersetzt und erläutert*. Quellenschriften für Kunstgeschichte und Kunsttechnik des Mittelalters und der Neuzeit, Neue Folge, II. Band. Vienna: Carl Graeser, 1889. URL: http://www.literature.at/alo?objid=16478 (visited on 17/7/2017).

**Wolff 1716**
Christian Wolff. *Mathematisches Lexicon, darinnen die in allen Theilen der Mathematick üblichen Kunst-Wörter erkläret, und zur Historie der Mathematischen Wissenschaften dienliche Nachrichten ertheilet, auch die Schrifften, wo iede Materie ausgeführet zu finden, angeführet werden*. Leipzig: Joh. Friedrich Gleditschens seel. Sohn, 1716. URL: http://www.mdz-nbn-resolving.de/urn/resolver.pl?urn=urn:nbn:de:bvb:12-bsb10594421-2 (visited on 17/7/2017).

**Zilsel 1942**
Edgar Zilsel. "The Sociological Roots of Science". *American Journal of Sociology* 47.4 (1942), 544–562. DOI: 10.1086/218962.

**JENS HØYRUP**

Dr. phil. (1996), cand. scient. (Copenhagen 1969), is emeritus professor at the Section for Philosophy and Science Studies at Roskilde University, and Honorary Research Fellow at the Chinese Academy of Sciences Institute for the History of Natural Science. His research includes the history of mathematics in pre- and early modern cultures; secondary interest in social and societal aspects of modern science. Also publications in linguistics and in the philosophy of science.

Jens Høyrup
Roskilde University
Section for Philosophy and Science Studies
P.O. Box 260
4000 Roskilde, Denmark
E-mail: jensh@ruc.dk